NASA TECHNICAL MEMORANDUM

NASA TM X-53644

August 8, 1967

NASA TM X-53644

AN ALGORITHM FOR THE DETERMINATION OF THE DEFINITENESS OF A REAL SQUARE MATRIX

By Russell M. Hayes, Jr.
Aero-Astrodynamics Laboratory

NASA

George C. Marshall Space Flight Center, Huntsville, Alabama

TECHNICAL MEMORANDUM X-53644

AN ALGORITHM FOR THE DETERMINATION OF THE DEFINITENESS OF A REAL SQUARE MATRIX

By

Russell M. Hayes, Jr.

George C. Marshall Space Flight Center

Huntsville, Alabama

ABSTRACT

An algorithm is described whereby the definiteness property of a real square matrix may be easily computed. The technique is based on well known theorems of matrix algebra which have been modified slightly to permit easy manual or machine computation.

A FORTRAN program using the algorithm was written and was run on both the IBM 1130 and the SDS 930 digital computers. The computation time for a 7 x 7 matrix was approximately one second on the IBM 1130 and much less than one second on the SDS 930.

NASA - GEORGE C. MARSHALL SPACE FLIGHT CENTER

Technical Memorandum X-53644

August 8, 1967

AN ALGORITHM FOR THE DETERMINATION OF THE DEFINITENESS OF A REAL SQUARE MATRIX

By

Russell M. Hayes, Jr.

OPTIMIZATION THEORY BRANCH
ASTRODYNAMICS AND GUIDANCE THEORY DIVISION
AERO-ASTRODYNAMICS LABORATORY
RESEARCH AND DEVELOPMENT OPERATIONS

TABLE OF CONTENTS

TECHNICAL MEMORANDUM X-53644

AN ALGORITHM FOR THE DETERMINATION OF THE DEFINITENESS OF A REAL SQUARE MATRIX

SUMMARY

An algorithm is described whereby the definiteness property of a real square matrix may be easily computed. The technique is based on well known theorems of matrix algebra which have been modified slightly to permit easy manual or machine computation.

A FORTRAN program using the algorithm was written and was run on both the IBM 1130 and the SDS 930 digital computers. The computation time for a 7 x 7 matrix was approximately one second on the IBM 1130 and much less than one second on the SDS 930.

I. INTRODUCTION

Many engineering problems require the determination of that property of a real square matrix known as its definiteness. Such problems include Lyapunov stability theory in which the definiteness of both the V function and its time derivative are required, the determination of the positive definiteness of the solution of the matrix Riccati equation from optimal control theory, and the determination of the definiteness of network functions in the realm of network synthesis.

The definiteness property of a real square matrix is actually a property of the associated quadratic form; however, as the quadratic form is uniquely specified by the matrix, we ascribe definiteness to either the form or the matrix.

Most often we find the necessary and sufficient conditions for the various classes of definiteness to be stated as follows:

(1) A matrix is positive definite if and only if all of its leading principal minors are positive.

(2) A matrix is positive semi-definite if and only if all of its principal minors are non-negative.

(3) A matrix is negative definite if and only if all of its leading principal minors are non-zero and alternate in sign, the first order minor being negative. Alternatively, a matrix is negative definite if the negative of the matrix is positive definite.

(4) A matrix is negative semi-definite if and only if all principal minors which are non-zero alternate in sign; i.e., the first order minors are all negative, the second order minors are all positive, etc. Alternatively, a matrix is negative semi-definite if the negative of the matrix is positive semi-definite.

(5) A matrix which does not belong to one of the above categories is called indefinite.

We propose that a computationally more usable definition is based on the definiteness of a diagonal matrix congruent to the symmetric equivalent of the original matrix. The computation necessary to determine the congruent form is approximately the same as that required to evaluate a determinant of order equal to the rank of the original matrix. Once the diagonal matrix is computed, the definiteness property is ascertained by inspection of the signs of the diagonal terms.

The remainder of this paper is divided into three major sections. The first section is a non-formal proof of the technique used, the second describes the computational algorithm, and the last section is an appendix which includes a worked example and a FORTRAN program written for the SDS 930 digital computer. The FORTRAN program will determine the definiteness of any real square matrix up to order 10 x 10 as presently written and may be easily converted into a subroutine for use with other programs.

II. PROOF OF THE TECHNIQUE

We wish to show that, given any real square symmetric matrix $\underline{A}$ having a quadratic form defined by

$$Q = \langle \underline{x}, \underline{A}\,\underline{x} \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i\, x_j \tag{1}$$

where

$\underline{x}$ is an nth order column vector of variables

$\underline{A}$ is an $n \times n$ real square symmetric matrix.

we may determine a diagonal matrix $\underline{D}$, related to $\underline{A}$ by a non-singular transformation, which has a quadratic form covering the same range of values. This matrix will be shown to be unique and to have the form

$$\underline{D} = \left| \begin{array}{c:c:c} \underline{I}_q & \underline{O}_{q \times (r-q)} & \underline{O}_{q \times (n-r)} \\ \hdashline \underline{O}_{(r-q) \times q} & -\underline{I}_{r-q} & \underline{O}_{(r-q) \times (n-r)} \\ \hdashline \underline{O}_{(n-r) \times q} & \underline{O}_{(n-r) \times (r-q)} & \underline{O}_{(n-r) \times (n-r)} \end{array} \right| \tag{2}$$

where

n is the order of the matrix $\underline{A}$

r is the rank of the matrix $\underline{A}$

q is the index of the matrix $\underline{A}$

$\underline{I}_q$ is a $q \times q$ identity matrix

$\underline{I}_{r-q}$ is an $(r - q) \times (r - q)$ identity matrix

$\underline{O}_{i \times j}$ is an $i \times j$ null matrix.

We may then examine the quadratic form of $\underline{D}$ which appears as

$$Q^* = < \underline{y}, \underline{D}\ y > = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}\ x_i\ x_j \tag{3}$$

for its definiteness instead of equation (1). The definiteness of (3) is readily determined by inspection. We first expand (3) into

$$Q^* = y^2 + y_2^2 + \dots + y_q^2 - y_{q+1}^2 - y_{q+2}^2 - \dots - y_r^2. \tag{4}$$

We may now see that the following conditions are true:

(1) Q^* is positive definite if and only if $q = r = n$.

(2) Q^* is positive semi-definite if and only if $q = r < n$.

(3) Q^* is negative definite if and only if $q = 0$ and $r = n$.

(4) Q^* is negative semi-definite if and only if $q = 0$ and $r < n$.

(5) Q^* is indefinite if $q \neq 0$ and $q \neq r$.

Now since the form of (4) covers the same range of values as that of (1), the statements as to the definiteness of (4) hold also for (1).

Our problem now becomes one of showing that such a $\underline{D}$ matrix exists for all real square $\underline{A}$ matrices and that it is unique; i.e., q and r are unique. To do this we will divide our proof into four parts. In the first we will show that the quadratic forms of two matrices related by a non-singular transformation cover the same range of values. Next we show that the quadratic form of any real square matrix may be considered to be the quadratic form of a real symmetric matrix of the same order, and thus, without loss of generality, we may consider our matrix to be symmetric in all cases. In the third part we will show that, given a symmetric matrix $\underline{A}$, we may always find the diagonal form of equation (2) by a non-singular transformation on $\underline{A}$, e.g., $\underline{D} = \underline{P}' \, \underline{A} \, \underline{P}$ where P is non-singular and ' denotes the transpose. Finally, we prove that the $\underline{D}$ so obtained is unique.

A. Quadratic Forms of Matrices Related by Non-Singular Transformations

In this section we wish to show that two matrices related by a non-singular transformation of the form

$$\underline{B} = \underline{P}' \, \underline{A} \, \underline{P}, \tag{5}$$

where $\underline{P}$ is an $n \times n$ non-singular matrix, have quadratic forms covering the same range of values. We first consider the quadratic form of $\underline{A}$,

$$Q = \langle \underline{x}, \underline{A} \, \underline{x} \rangle . \tag{6}$$

Next, consider the following variable transformation:

$$\underline{x} = \underline{P}\ \underline{y} \tag{7}$$

where

$\underline{P}$ is an n $\times$ n non-singular matrix.

Substitution of (7) into (6) yields

$$Q = \langle \underline{P}\ \underline{y},\ \underline{A}\ \underline{P}\ \underline{y} \rangle = \langle \underline{y},\ \underline{P}'\ \underline{A}\ \underline{P}\ \underline{y} \rangle , \tag{8}$$

which by (5) is the quadratic form of $\underline{B}$:

$$Q = \langle \underline{y},\ \underline{B}\ \underline{y} \rangle . \tag{9}$$

Now, for any $\underline{y}$ yielding a value of $Q = Q_1$ in (9), we may obtain a unique $\underline{x}$ from (7) yielding the same value of $Q = Q_1$ from (6). Therefore, the quadratic forms of two matrices which are related by the transformation of (5) cover identical ranges.

B. Equivalent Symmetric Matrix

In this section we show that the quadratic form of any real square matrix, $\underline{A}$ may be considered to be the quadratic form of a real symmetric matrix $\underline{S}$ related to $\underline{A}$ by

$$\underline{S} = \frac{1}{2} (\underline{A} + \underline{A}') \tag{10}$$

where

$\underline{S}$ is a symmetric matrix of nth order.

Equation (10) may also be written

$$(s_{ij}) = \frac{1}{2} (a_{ij} + a_{ji}), \tag{11}$$

where

S_{ij} is an element of $\underline{S}$

a_{ij} is an element of $\underline{A}$.

Consider the quadratic form of $\underline{A}$.

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j. \tag{12}$$

We may write this as

$$Q = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j. \tag{13}$$

It is merely a matter of notation to interchange the subscripts of the second term on the right-hand side of (13) to obtain

$$Q = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ji} x_j x_i, \tag{14}$$

which is the same as

$$Q = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ji} x_i x_j \tag{15}$$

or

$$Q = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} + a_{ji}) x_i x_j. \tag{16}$$

Substitution of (11) into (16) yields

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} S_{ij}\, x_i\, x_j . \tag{17}$$

Therefore, the quadratic forms of (12) and (17) are identical. Considering this, we see that we may always assume our $\underline{A}$ matrix to be symmetric without loss of generality. If we wish to apply the techniques developed in the following sections to a non-symmetric matrix, we simply obtain the equivalent symmetric form by equation (10) and proceed.

C. Determination of Diagonal Matrix $\underline{D}$

Now, we wish to show that, given a symmetric matrix $\underline{A}$, we may always obtain a diagonal matrix $\underline{D}$ as defined by equation (2) through a non-singular transformation.

In constructing our proof, we shall require use of the following three elementary operations:

(1) interchange of two columns (rows),

(2) multiplication of a column (row) by a constant,

(3) multiplication of a column (row) by a constant followed by addition of that column (row) to another column (row).

The column operations may be performed by postmultiplying the matrix $\underline{A}$ by one of the following defined elementary transformation matrices:

$$\underline{E}_1 = \begin{array}{cc} & \begin{array}{ccccc} & i & & j & \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \left[\begin{array}{c:c:c:c:c} \underline{I} & & & & \\ \hdashline & 0 & & 1 & \\ \hdashline & & \underline{I} & & \\ \hdashline & 1 & & 0 & \\ \hdashline & & & & \underline{I} \end{array}\right] \end{array}, \quad \det \underline{E}_1 = -1$$

(equation (18) continued on next page)

$$\underline{E}_2 = \begin{array}{cc} & \begin{array}{ccc} \quad & k & \quad \end{array} \\ k & \left[\begin{array}{c:c:c} \underline{I} & & \\ \hdashline & e_{kk} & \\ \hdashline & & \underline{I} \end{array}\right] \end{array}, \qquad \det \underline{E}_2 = e_{kk} \tag{18}$$

$$\underline{E}_3 = \begin{array}{cc} & \begin{array}{ccccc} \quad & i & \quad & j & \quad \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \left[\begin{array}{c:c:c:c:c} \underline{I} & & & & \\ \hdashline & 1 & & e_{ij} & \\ \hdashline & & \underline{I} & & \\ \hdashline & & & 1 & \\ \hdashline & & & & \underline{I} \end{array}\right] \end{array}, \qquad \det \underline{E}_3 = 1.$$

Postmultiplication of $\underline{A}$ by one of the elementary transformation matrices performs a column operation, while premultiplication of $\underline{A}$ by the transpose of the elementary transformation matrix results in the equivalent row operation. All of these transformation matrixes are obviously non-singular, and any matrix formed from a product of such non-singular matrices is also non-singular. That is, the determinant of such a product is equal to the product of the determinants which, being non-zero, indicates that the product matrix is non-singular.

We will now show that a symmetric matrix $\underline{A}$ may be reduced to the form of $\underline{D}$. Consider first that the matrix $\underline{A}$ has a non-zero diagonal element. This element is moved to the a_{11} location by a column interchange and a row interchange. Of course, if the a_{11} location is non-zero initially, the row and column interchanges are unnecessary. We next multiply column one and row one by $1/\sqrt{|a_{11}|}$. This operation results in a plus one or a minus one in the a_{11} location. If the first row is next multiplied by $-a_{21}/+\sqrt{a_{11}}$ and added to the second, we may obtain a zero in the a_{21} location. Similarly, multiplication of the first column by the same value and addition of this column to the second results in a zero in the a_{12} location. Proceeding in a similar fashion, we may zero all of the elements of the first row and of the first column except for the a_{11} location which remains at plus or minus one. Our matrix is now of the form

$$\underline{R}'\ \underline{A}\ \underline{R} = \left[\begin{array}{c|ccc} \pm 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \underline{C} & \\ 0 & & & \end{array}\right], \tag{19}$$

where $\underline{R}$ is the product of the required elementary transformation matrices and $\underline{C}$ is an $(n - 1) \times (n - 1)$ reduced matrix.

The procedure outlined above may be repeated to reduce $\underline{C}$, and so on, until a diagonal form is attained.

A difficulty which may be encountered in this process is the case where all diagonal elements are zero at some point in the reduction, but there are non-zero off-diagonal elements. This case may be handled in the following manner. Assume that a non-zero off-diagonal element is found at a_{ij}. Of course, since the matrix is symmetric, there will be an identical element at a_{ji}. We may add column j to column i and then add row j to row i to form a main diagonal element at a_{ij} which has the value $a_{ij} + a_{ji} = 2a_{ij}$. We may now proceed as before. This is the technique used in the program as written so that a form as defined by (2) is obtained. It may be shown that a matrix having an element at a_{ij} but no element at a_{jj} is indefinite. This may be seen if we assume that we set all of the elements of $\underline{x}$ equal to zero except for x_i and x_j. The quadratic form will become

$$Q = a_{ii}\, x_i^2 + 2a_{ij}\, x_i\, x_j \tag{20}$$

from which it is obvious that given any x_i we may always choose a value of x_j to make Q either positive or negative as we wish. Therefore, if we encounter this characteristic at any point in the reduction the matrix is indefinite.

In case we find no diagonal nor off-diagonal non-zero elements, our procedure is terminated as we have obtained the desired congruent form.

From the foregoing discussion, we can see that a diagonal form so defined by (2) may be obtained for all cases. We must next determine whether or not the form so attained is unique; that is, does it have the same rank and index regardless of the sequence of elementary operations used. We will consider this problem in two parts. First we will show that the rank so attained is unique and second that the index is also unique.

D. Proof of the Uniqueness of $\underline{D}$

We next show that the rank of $\underline{D}$ is equal to the rank of $\underline{A}$ under the congruent transformation $\underline{D} = \underline{P}'\ \underline{A}\ \underline{P}$. Let the matrix $\underline{A}$ be written

$$\underline{A} = \big| a_1\ a_2 \ldots a_r (c_{11}a_1 + c_{21}a_2 + \ldots\ c_{r1}a_r) \ldots$$

$$\ldots (c_{1(n-r)}a_1 + c_{2(n-r)}a_2 + \ldots + c_{r(n-r)}a_r)\big| \tag{21}$$

where a_i is one of r independent columns of $\underline{A}$ and the c_{ij}'s are p constants used to form the remaining (n - r) dependent columns of $\underline{A}$.

Consider the product,

$$\underline{P}'\ \underline{A} = \underline{B} = \left|\begin{array}{l}
\langle p_1,a_1\rangle\ \langle p_1,a_2\rangle \ldots \langle p_1,a_r\rangle\ \langle p_1,(c_{11}a_1+c_{21}a_2+\ldots+c_{r1}a_r)\rangle \\
\qquad\qquad \ldots\ p_1,(c_{1(n-r)}a_1+c_{2(n-r)}a_2+\ldots+c_{r(n-r)}a_r)\rangle \\
\\
\langle p_2,a_1\rangle\ \langle p_2,a_2\rangle \ldots \langle p_2,a_r\rangle\ \langle p_2,(c_{11}a_1+c_{21}a_2+\ldots+c_{r1}a_r)\rangle \\
\qquad\qquad \ldots \langle p_2,(c_{1(n-r)}a_1+c_{2(n-r)}a_2+\ldots+c_{r(n-r)}a_r)\rangle \\
- \\
\langle p_n,a_1\rangle\ \langle p_n,a_2\rangle \ldots \langle p_n,a_r\rangle\ \langle p_n,(c_{11}a_1+c_{21}a_2+\ldots+c_{r1}a_r)\rangle \\
\qquad\qquad \ldots \langle p_n,(c_{1(n-r)}a_1+c_{2(n-r)}a_2+\ldots+c_{r(n-r)}a_r)\rangle
\end{array}\right|$$

(22)

where the p_i's are columns of $\underline{P}$. Expanding the n - r columns on the right our matrix becomes

$$\underline{B} = \left| \begin{array}{l} \langle p_1,a_1\rangle \; \langle p_1,a_2\rangle \cdots \langle p_1,a_r\rangle \; c_{11}\langle p_1,a_1\rangle + c_{21}\langle p_1,a_2\rangle + \dots + c_{r1}\langle p_1,a_r\rangle \\ \qquad\qquad\qquad \cdots \; c_{1n-r}\langle p_1,a_1\rangle + c_{2n-r}\langle p_1,a_2\rangle + \dots + c_{rn-r}\langle p_1,a_r\rangle \\ \langle p_2,a_1\rangle \; \langle p_2,a_2\rangle \cdots \langle p_2,a_r\rangle \; c_{11}\langle p_2,a_1\rangle + c_{21}\langle p_2,a_2\rangle + \dots + c_{r1}\langle p_2,a_r\rangle \\ \qquad\qquad\qquad \cdots \; c_{1n-r}\langle p_2,a_1\rangle + c_{2n-r}\langle p_2,a_2\rangle + \dots + c_{rn-r}\langle p_2,a_r\rangle \\ - \\ \langle p_n,a_1\rangle \; \langle p_n,a_2\rangle \cdots \langle p_n,a_r\rangle \; c_{11}\langle p_n,a_1\rangle + c_{21}\langle p_n,a_2\rangle + \dots + c_{r1}\langle p_n,a_r\rangle \\ \qquad\qquad\qquad \cdots \; c_{1n-r}\langle p_n,a_1\rangle + c_{2n-r}\langle p_n,a_2\rangle + \dots + c_{rn-r}\langle p_n,a_r\rangle \end{array} \right| \tag{23}$$

from which it is obvious that the rank of $\underline{P}'\,\underline{A}$ is at most r, the rank of $\underline{A}$.

Now, since $\underline{P}'$ is non-singular, we may write

$$\underline{A} = \underline{P}'^{-1}\,\underline{B}, \tag{24}$$

where

$$\underline{B} = \underline{P}'\,\underline{A}.$$

Following the procedure outlined above, we can see that the rank of $\underline{A}$ is at most that of $\underline{B}$ and thus the rank of $\underline{B}$ must equal that of $\underline{A}$.

The same procedure may also be followed to show that the product $\underline{P}'\,\underline{A}\,\underline{P} = (\underline{P}'\,\underline{A})\,\underline{P} = \underline{B}\,\underline{P}$ has the rank of $\underline{B}$ or, from our previous proof, the rank of $\underline{A}$.

To conclude the proof of uniqueness, we need to show that the index of $\underline{D}$ is unique regardless of the sequence of elementary transformations used to obtain $\underline{D}$. Consider that there are two transformations on $\underline{A}$ yielding two matrices, $\underline{D}_1$ and $\underline{D}_2$, having the same rank but different indices. That is,

$$\underline{P}' \; \underline{A} \; \underline{P} = \underline{D}_1 \tag{25}$$

and

$$\underline{R}' \; \underline{A} \; \underline{R} = \underline{D}_2,$$

where $\underline{P}$ and $\underline{R}$ are non-singular transformation matrices.

Now let

$$\underline{x} = \underline{P} \; \underline{y}$$

and (26)

$$\underline{x} = \underline{R} \; \underline{z}.$$

For the first case our quadratic form becomes

$$Q = < \underline{x}, \; \underline{A} \; \underline{x} > = < \underline{y} \; \underline{P}, \; \underline{A} \; \underline{P} \; \underline{y} > = < \underline{y}, \; \underline{P}' \; \underline{A} \; \underline{P} \; y > = < \underline{y}, \; \underline{D}_1 \; \underline{y} > \tag{27}$$

and for the second it becomes

$$Q = < \underline{x}, \; \underline{A} \; \underline{x} > = < \underline{z} \; \underline{R}, \; \underline{A} \; \underline{R} \; \underline{y} > = < \underline{z}, \; \underline{R}' \; \underline{A} \; \underline{R} \; \underline{z} > = < \underline{z}, \; \underline{D}_2 \; \underline{z} > . \tag{28}$$

We will assume that the index of $\underline{D}_1$ is equal to q and that the index of $\underline{D}_2$ is equal to p and $q \neq p$. Our quadratic forms given by (27) and (28) may be rewritten as

$$q = y_1^2 + y_2^2 + \ldots + y_q^2 - y_{q+1}^2 - \ldots - y_r^2 \tag{29}$$

and

$$Q = z_1^2 + z_2^2 + \ldots + z_p^2 - z_{p+1}^2 - \ldots - z_r^2 . \tag{30}$$

It is simply a matter of notation to assume that $q < p$. We now write the following homogeneous set of equations:

$$\begin{aligned} y_i &= 0 \qquad & i &= 1,2, \ldots, q \\ z_j &= 0 \qquad & j &= p+1, \ldots, n . \end{aligned} \tag{31}$$

We have a set of $q + n - p < n$ equations in n unknown x_i's, and thus a non-trivial solution exists. That is, $\underline{x} = \underline{x}^* \neq 0$ is a solution of (31). We may also compute

$$\begin{aligned} \underline{y}^* &= \underline{P}^{-1}\, \underline{x}^* \\ \underline{z}^* &= \underline{R}^{-1}\, \underline{x}^*. \end{aligned} \tag{32}$$

From (29) and (31) $Q \leq 0$ but from (30) and (31) $Q \geq 0$ and thus $Q = 0$. Observing (27) and (28), we see that this is only possible if $y^* = z^* = 0$, but from (32) and the fact that a non-trivial solution to (31) exists, we see that this is not the case.

The foregoing contradictions arise from our assumption that $p < q$ and since we arrive at the same contradictions if we assume $q < p$, we see that $p = q$ and $\underline{D}_1 = \underline{D}_2$. Therefore, our congruent form is unique, and the sequence of application of elementary transform pairs is immaterial.

III. COMPUTATIONAL ALGORITHM

The congruent diagonal matrix $\underline{D}$ may be computed without developing the transformation matrices directly. The procedure used will be the following:

<u>Step 1</u>. We form the equivalent symmetric matrix from our given matrix using the algorithm

$$S_{ij} = \frac{1}{2}(a_{ij} + a_{ji}), \tag{33}$$

where S_{ij} is an element of the symmetrized matrix.

NOTE: In the remainder of the algorithm elements of all matrices will be referred to by a_{ij} rather than S_{ij}.

Step 2. The main diagonal is searched for the element with the largest magnitude. This element is moved to the a_{11} location by a row and column interchange. We work with the largest element to reduce as much as possible numerical computational errors. If a main diagonal element is zero, then off-diagonal locations are searched for a non-zero element. If we do find a non-zero element, then the matrix is indefinite as was shown previously; however, we continue to obtain the matrix $\underline{D}$. Let us assume that the non-zero element is located at a_{ij}; then, we add column j to column i and row j to row i to form a non-zero $a_{ii} = 2a_{ij}$.

If all main diagonal elements are zero and all off-diagonal elements are also zero, then we have obtained the desired form of $\underline{D}$ and the procedure is halted.

Assuming that a non-zero a_{11} has been obtained, we divide the first row by the magnitude of that element; i.e.,

$$a_{1j} = a_{1j}/|a_{11}| \qquad j = 1, \ldots, n. \tag{34}$$

Step 3. Next, we apply the following algorithm to the remaining n - 1 rows:

$$a_{ij} = a_{ij} - a_{i1} \cdot a_{11} \cdot a_{1j} \qquad \begin{array}{l} i = 2, \ldots, n \\ j = 1, \ldots, i. \end{array} \tag{35}$$

This zeroes all of the first column except for a_{11} which remains ± 1 and modifies the remainder of the lower triangular portion of the matrix.

Step 4. The matrix is next resymmetrized by applying the following:

$$a_{ij} = a_{ji} \qquad \begin{array}{l} i = 1, \ldots, n - 1 \\ j = i + 1, \ldots, n. \end{array} \tag{36}$$

The preceding four steps are equivalent to applying several elementary transformations, and at the end we are left with a matrix such as that of (19).

Next, we recycle through steps 2, 3, and 4 as applied to the reduced matrix ($\underline{C}$ of equation (19)). The process is continued until the matrix $\underline{D}$ is obtained. Our matrix $\underline{D}$ will not necessarily have all of the +1's grouped together and all of the -1's grouped together as indicated by (2), but this, while unnecessary, could be affected by row-row and column-column interchanges.

Step 5. The final procedure is to search the main diagonal of $\underline{D}$, counting the number of +1's and the number of -1's. The following rules are then applied to obtain the definiteness of the matrix $\underline{D}$ and, through it, the definiteness of the matrix $\underline{A}$.

(1) If the number of +1's is equal to the order, n, of $\underline{A}$, then the matrix is positive definite.

(2) If the number of +1's is less than the order, n, and there are no -1's, then the matrix is positive semi-definite. This also includes the case where there are no +1's.

(3) If the number of -1's is equal to the order, n, of $\underline{A}$ then the matrix is negative definite.

(4) If the number of -1's is less than the order, n, and there are no +1's, then the matrix is negative semi-definite.

(5) If there are both +1's and -1's present, then the matrix is indefinite.

IV. CONCLUSIONS

1. We have proven an algorithm for the computation of the definiteness of a real square matrix. While the theorems of matrix algebra used in the proof are not new, it is believed that the means of obtaining the matrix $\underline{D}$ computationally is new and that the technique offers significant advantages over the commonly used definition of definiteness as was given in the introduction to this paper.

2. The algorithm described above has been programmed in FORTRAN (see appendix B) and run on both the SDS 930 and the IBM 1130 computer. A seven-by-seven positive definite matrix was run on each machine

(a definite matrix was used because this required the maximum computation for a given matrix size). On the IBM 1130, the computation time was approximately one second, while on the SDS 930, the output ran at the maximum printer speed of 640 lines per minute, and thus no computation times could be measured. However, the time was much less than one second.

3. Since this technique requires computation roughly equivalent to that of evaluating a determinant equal to the rank of the matrix, it should be much faster than any procedure based on successive computation of minors.

4. In addition to determining the definiteness of a matrix, the congruent form gives the rank of the matrix and also the number of positive and negative eigenvalues of the symmetric form. The number of positive and negative eigenvalues corresponds, respectively, to the number of positive and negative entries on the main diagonal of the congruent diagonal form. This may be limited in usefulness, however, since there is no information given about the eigenvalues of the original non-symmetric matrix.

APPENDIX A

Example Problem

Consider the three-by-three matrix

$$A = \begin{vmatrix} -5 & -14 & 6 \\ 0 & -13 & 10 \\ -12 & 0 & -2 \end{vmatrix} . \tag{37}$$

Step 1. Symmetrizing by equation (33),

$$A_{sym} = \begin{vmatrix} -5 & -7 & -3 \\ -7 & -13 & -5 \\ -3 & -5 & -2 \end{vmatrix} . \tag{38}$$

Step 2. Applying equation (34),

$$\begin{vmatrix} -1 & -7/5 & -3/5 \\ -7 & -13 & -5 \\ -3 & -5 & -2 \end{vmatrix} . \tag{39}$$

Step 3. Applying equation (35) to rows two and three,

$$\begin{vmatrix} -1 & -7/5 & -3/5 \\ 0 & -16/5 & -5 \\ 0 & -4/5 & -1/5 \end{vmatrix} . \tag{40}$$

Step 4. Applying equation (36),

$$\begin{vmatrix} -1 & 0 & 0 \\ 0 & -16/5 & -4/5 \\ 0 & -4/5 & -1/5 \end{vmatrix}. \tag{41}$$

Reapplying steps 2 through 4 to the reduced (two-by-two) matrix,

Step 2. $$\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & -1/4 \\ 0 & -4/5 & -1/5 \end{vmatrix}. \tag{42}$$

Step 3. $$\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & -1/4 \\ 0 & 0 & 0 \end{vmatrix}. \tag{43}$$

Step 4. $$\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}. \tag{44}$$

Step 5. Equation (44) is the desired congruent matrix. Since it is obviously negative semi-definite, the matrix given by equation (37) is negative semi-definite. Here, we have omitted the main diagonal search for the largest element since this was not required to illustrate the method.

APPENDIX B

FORTRAN Program Description

Following is listed the FORTRAN program as written for the SDS 930 digital computer. No claim is made that this program is optimally fast; however, it does run at a speed which should be satisfactory for most applications.

The program requires as input the order of the matrix and the elements of the matrix entered row-wise. The output consists of a listing of the original matrix, the symmetrized matrix, the diagonal congruent form, and a comment as to the definiteness of the original matrix. Occasionally, an output statement "A DIFFERENCE LESS THAN 1.0E-06 WAS ENCOUNTERED" occurs. This statement implies that, during the computation, a difference between two numbers was obtained which was less than 1.0×10^{-4} percent of the magnitude of one of the numbers. When this happens, the program considers this difference to be zero and thus stores zero instead of the computed difference. This helps to eliminate some errors from truncation, but it is conceivable that such a small difference would exist. Therefore, problems for which this occurs should be checked carefully.

The data required for the program should be entered as follows:

(1) The first data card contains the order of the matrix punched in columns one and two, right justified (FORMAT (I2)). The maximum size presently allowed is a tenth order matrix.

(2) The remaining data cards contain the elements of the matrix, entered row-wise, with eight entries per card, each in a field ten columns wide. The decimals must be punched (FORMAT (8F10.0)).

Upon completion of the run, the program returns to the first read statement in order to read data for a second computation.

```
ΔASSIGN S=MT0,SI=CR,B0=MT1,L0=LP.
ΔREWIND MT1.
ΔFORTRAN B0,L0.
=     C
=     C THIS PROGRAM DETERMINES THE DEFINITENESS OF A REAL SQUARE MATRIX
=     C
=           DIMENSION A[10,10]
=           MM = 2
=           NN = 3
=     C
=     C READ IN REAL MATRIX
=     C
=       802 K6 = 1
=           READ 1,N
=         1 FORMAT [I2]
=           READ 2, [[A[I,J],J=1,N],I=1,N]
=         2 FORMAT [8F10.0]
=     C
=     C WRITE ORIGINAL MATRIX
=     C
=           PRINT 3
=         3 FORMAT [16H1ORIGINAL MATRIX/]
=           DO 100 I=1,N
=       100 PRINT 4, [A[I,J],J=1,N]
=         4 FORMAT [8E15.7]
=     C
=     C DETERMINE EQUIVALENT SYMMETRIC FORM
=     C
=           DO 101 I=1,N
=           DO 101 J=1,N
=           A[I,J] = [A[I,J] + A[J,I]]/2
=       101 A[J,I] = A[I,J]
=     C
=     C WRITE EQUIVALENT SYMMETRIC FORM
=     C
=           PRINT 5
=         5 FORMAT [17H0SYMMETRIC MATRIX/]
=           DO 114 I=1,N
=       114 PRINT 4, [A[I,J],J=1,N]
=     C
=     C DETERMINE CONGRUENT FORM
=     C
=           DO 102 K=1,N
=     C
=     C LOCATE LARGEST DIAGONAL ELEMENT
=     C
=           BIG = ABS[A[K,K]]
=           KK = K
=           DO 103 I=K,N
=           IF [A[I,I]] 135,130,135
=     C
=     C IF A ZERO IS ON THE MAIN DIAGONAL CHECK ROW FOR NON-ZERO ELEMENTS
=     C
```

```
  130 DO 131 J=K,N
      IF [A[I,J]] 132,131,132
  131 CONTINUE
      GO TO 135
  132 DO 133 L=K,N
  133 A[I,L] = A[I,L] + A[J,L]
      DO 134 L=K,N
  134 A[L,I] = A[L,I] + A[L,J]
  135 IF [ABS[A[I,I]] - BIG] 103,103,104
  104 BIG = ABS[A[I,I]]
      KK = I
  103 CONTINUE
C
C IF LARGEST ELEMENT IS ZERO, GO TO END ROUTINE
C
      IF [BIG] 109,110,109
C
C CHECK FOR NECESSITY OF INTERCHANGE
C
  109 IF [KK - K] 105,106,105
C
C INTERCHANGE ROWS AND COLUMNS
C
  105 DO 107 J=K,N
      TEMP = A[KK,J]
      A[KK,J] = A[K,J]
  107 A[K,J] = TEMP
      DO 108 I=K,N
      TEMP = A[I,KK]
      A[I,KK] = A[I,K]
  108 A[I,K] = TEMP
C
C DIVIDE WORKING ROW BY MAGNITUDE OF LEADING COEFFICIENT
C
  106 D = ABS[A[K,K]]
      DO 111 J=K,N
  111 A[K,J] = A[K,J]/D
C
C CHECK FOR COMPLETION
C
      IF [K - N] 112,110,112
C
C NOT FINISHED, ANIHILATE WORKING COLUMN
C
  112 KK = K + 1
      DO 113 I=KK,N
      DO 113 JJ=K,I
      J = I + K - JJ
      D = A[I,J]
      A[I,J] = A[I,J] - A[I,K]*A[K,K]*A[K,J]
C
C IF THE DIFFERENCE COMPUTED ABOVE IS LESS THAN ONE IN THE
C SIXTH SIGNIFICANT PLACE, SET EQUAL TO ZERO AND NOTE
C THIS IN THE OUTPUT
```

```
C
      IF [A[I,J]] 804,113,804
  804 IF [ABS[A[I,J]/D] - 1.0E-06] 801,801,113
  801 A[I,J] = 0.0
      K6 = 2
  113 CONTINUE
C
C FORM NEW SYMMETRIC MATRIX
C
      DO 102 J=KK,N
      DO 102 I=K,J
  102 A[I,J] = A[J,I]
C
C WRITE CONGRUENT FORM
C
  110 PRINT 6
    6 FORMAT [17HOCONGRUENT MATRIX/]
      DO 115 I=1,N
  115 PRINT 4, [A[I,J],J=1,N]
C
C DETERMINE DEFINITENESS OF THE CONGRUENT FORM AND PRINT
C
      K1 = 1
      K2 = 1
      K3 = 1
      K4 = 1
      K5 = 1
      DO 116 K=1,N
      IF [A[K,K]] 117,118,119
  119 K3 = 2
      GO TO 116
  118 K4 = 2
      K5 = 2
      GO TO 116
  117 K1 = 2
      K2 = 2
  116 CONTINUE
      GO TO [120,121],K1
  120 GO TO [122,123],K2
  122 GO TO [124,125],K4
  121 GO TO [126,123],K3
  126 GO TO [127,128],K5
  123 PRINT 7
      GO TO 129
  124 PRINT 8
      GO TO 129
  125 PRINT 9
      GO TO 129
  127 PRINT 10
      GO TO 129
  128 PRINT 11
      GO TO 129
  129 GO TO [802,803],K6
  803 PRINT 12
```

```
      GO TO 802
    7 FORMAT [26HOTHIS MATRIX IS INDEFINITE]
    8 FORMAT [33HOTHIS MATRIX IS POSITIVE DEFINITE]
    9 FORMAT [38HOTHIS MATRIX IS POSITIVE SEMI-DEFINITE]
   10 FORMAT [33HOTHIS MATRIX IS NEGATIVE DEFINITE]
   11 FORMAT [38HOTHIS MATRIX IS NEGATIVE SEMI-DEFINITE]
   12 FORMAT [1HO,$A DIFFERENCE LESS THAN 1.0E-06 WAS ENCOUNTERED$]
      END
```

```
PROGRAM ALLOCATION

  00013 A          00323 MM         00324 NN         00325 K6
  00326 N          00327 I          00330 J          00331 K
  00332 KK         00333 L          00334 JJ         00335 K1
  00336 K2         00337 K3         00340 K4         00341 K5
  00342 BIG        00344 TEMP       00346 D

SUBPROGRAMS REQUIRED

  ABS
THE END
```

REFERENCES

1. Perlis, Sam, "Theory of Matrices," Addison-Wesley Publishing Co., Inc., 1952.

2. Bellman, R., "Introduction to Matrix Analysis," McGraw-Hill Book Co., Inc., 1960.

APPROVAL NASA TM X-53644

AN ALGORITHM FOR THE DETERMINATION OF THE DEFINITENESS OF A REAL SQUARE MATRIX

by Russell M. Hayes, Jr.

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

J C Blair

James C. Blair
Chief, Optimization Theory Branch

for J C Blair

Clyde D. Baker
Chief, Astrodynamics and Guidance Theory Division

Geissler

E. D. Geissler
Director, Aero-Astrodynamics Laboratory

DISTRIBUTION

R-DIR, Mr. Weidner

R-ASTR, Dr. Haeussermann
Mr. Moore
Mr. Hosenthien

R-COMP-DIR

R-AERO-DIR, Dr. Geissler
R-AERO-DIR, Mr. Jean
R-AERO-D, Mr. Horn
R-AERO-D, Mr. Rheinfurth
R-AERO-D, Mr. Lewis
R-AERO-D, Mr. Ryan
R-AERO-D, Mr. Deaton
R-AERO-D, Mrs. Chandler
R-AERO-T, Mr. Cummings
R-AERO-T, Dr. Krause
R-AERO-T, Dr. Sperling
R-AERO-F, Mr. Lindberg
R-AERO-F, Mr. Benson
R-AERO-Y, W. Vaughan
R-AERO-Y, Mr. Dickey
R-AERO-X, Mr. Thomae
R-AERO-X, Mr. Wood
R-AERO-G, Mr. Baker
R-AERO-G, Mr. Dearman
R-AERO-G, Dr. Gunderson
R-AERO-G, Mr. Hill
R-AERO-G, Mr. Schwaniger
R-AERO-G, Mr. Causey
R-AERO-G, Mr. Blair
R-AERO-G, Mr. Redus
R-AERO-G, Mr. Ingram
R-AERO-G, Mr. Burrows
R-AERO-G, Mr. Heuser
R-AERO-G, Mr. Burns
R-AERO-G, Mr. Livingston
R-AERO-G, Mr. Hayes (10)

MS-IL (8)
MS-IP
MS-H
I-RM-M
CC-P
MS-T (6)
DEP-T
Scientific and Technical
Information Facility (25)
ATTN: NASA Representative
(S-AK/RKT)
P. O. Box 33
College Park, Maryland

Dr. L. B. Jenkins
Electrical Engineering Dept.
University of Louisville
Louisville, Kentucky 40208

Dr. Wai Kai Chen
Electrical Engineering Dept.
Ohio University
Athens, Ohio 45701

Dr. H. F. Klock
Electrical Engineering Dept.
Ohio University
Athens, Ohio 45701

Dr. J. L. Throne
Chemical Engineering Dept.
Ohio University
Athens, Ohio 45701

Dr. C. D. Johnson
University of Alabama
Research Center
301 Sparkman Drive
Huntsville, Alabama

Dr. Arthur Bennett
Boeing Company
Mail Stop AK-39
Huntsville, Alabama

DISTRIBUTION

Dr. Reynolds Beal
Manager, Dynamics & Guidance
Lockheed Missiles & Space Co.
4800 Bradford Drive
Huntsville, Alabama

Dr. E. G. Rynaski
Flight Research Department
Cornell Aeronautical Laboratory
P. O. Box 235
Buffalo, New York 14221

NASA, Ames Research Center
Moffett Field, California 94035
ATTN: Librarian

NASA, Electronics Research Center
575 Technology Square
Cambridge, Massachusetts 02139
ATTN: Librarian

Dr. D. K. Frederick
Electrical Engineering Dept.
Rensselaer Polytechnic Institute
Troy, New York 12181

Dr. S. Stephenson
Electrical Engineering Dept.
University of Arkansas
Fayetteville, Arkansas

Dr. D. Chesak
Electrical Engineering Department
Ohio University
Athens, Ohio 45701

Dr. J. Oliver
Engineering Mechanics Department
Louisiana State University
Baton Rouge, Louisiana

NASA, Goddard Space Flight Center
Greenbelt, Maryland 20771
ATTN: Librarian

NASA, Kennedy Space Center
Kennedy Space Center, Florida 32899
ATTN: Librarian

NASA, Lewis Research Center
21000 Brookpark Road
Cleveland, Ohio 44135
ATTN: Librarian

NASA, Manned Spacecraft Center
Houston, Texas 77058
ATTN: Librarian

Jet Propulsion Laboratory
4800 Oak Drive
Pasadena, California 91103
ATTN: Librarian

NASA, Langley Research Center
Langley Station
Hampton, Virginia 23365